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Energy Decays Locally Even if Total Energy Grows Algebraically with Time

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1. INTRODUCTION

We prove that energy decays locally as $T \to \infty (T = \text{time})$ at the rate T^{-2+k} for solutions of hyperbolic equations, with coefficients that depend upon both position and time, in the exterior of star-shaped domains in $\mathbb{R}^3$. Here k is a positive constant, depending on the coefficients, defined explicitly in Section 3 below. Our results generalize those of Zachmanoglou [7]. He considered a class of equations with time-independent coefficients [see (5.1) below]. and proved, under hypotheses roughly analogous to ours, that in $\mathbb{R}^n (n \geqslant 3)$ energy decays locally as $T^{-1+\mu} (1 > \mu \geqslant 0)$. A more significant difference than the difference in the rate of energy decay, between Zachmanoglou's results and ours is that we treat equations with solutions whose total energy may grow algebraically with T, while the total energy of solutions of the equations considered in [7] is conserved. In [1] we proved that the energy of solutions with bounded total energy decays locally as T^{-2}, but under more stringent hypotheses than those imposed here.

We now set the scattering problem whose solutions we investigate. Let V be the exterior of a closed, bounded subset B of $\mathbb{R}^3$, and let n be the interior normal to ∂V. We assume that the origin lies interior to B, and that ∂V is star-shaped,

$$\min_{x \in \partial V} \frac{n \cdot x}{r} \geqslant \sigma > 0$$

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where $x = (x_1, x_2, x_3)$ and $r^2 = x \cdot x$. Let

$$R = (V \cup \partial V) \times [0, \infty).$$

We use the notation

$$\nabla = [(\partial/\partial x_1), (\partial/\partial x_2), (\partial/\partial x_3)], \qquad \nabla^{(4)} = [\nabla, (\partial/\partial t)].$$

Let the transpose of a matrix M (or m) be M^T (or m^T). We take as given a symmetric 3×3 matrix E, 1×3 matrices a and b, and functions c and d, which satisfy the hypotheses:

(a) the elements of a, b and E, and the scalar functions c and d are bounded uniformly on R;

(b) b, c and E are in $C^1(R)$; a and d are in $C^2(R)$;

(c) for some $c_0 > 0$ and $d_0 > 0$, $\min_{|\xi|=1} \xi E(x, t)\, \xi^T \geqslant c_0$ and $d(x, t) \geqslant d_0$ for all (x, t) in R.

We adopt some notation to be used throughout. We define the matrices A and D by

$$A = \begin{pmatrix} E & a^T \\ a & -d \end{pmatrix}, \qquad D = \begin{pmatrix} E_t & b^T \\ b & -c \end{pmatrix}. \tag{1.1}$$

When a "dot" separates two vectors, it always indicates a scalar product. For example, $n \cdot x$ is $\sum_1^3 n_i x_i$ and $\nabla^{(4)} \cdot \alpha = \sum_1^4 (\partial\alpha^i/\partial x_i)$ if $\alpha = (\alpha^1, \alpha^2, \alpha^3, \alpha^4)$. When two vectors or matrices or a vector and a matrix are multiplied and no dot is present, the multiplication is always matrix multiplication. For example, $\nabla^{(4)}(\alpha^T) = \nabla^{(4)} \cdot \alpha$ if $\alpha = (\alpha^1,\ldots, \alpha^4)$ and $(\nabla^{(4)})^T\alpha$ is the matrix $(\alpha^j_{x_i})$. Moreover, $\nabla^{(4)}u$ means the vector grad u of a scalar function u; but if w is a column vector, then ∇w and ∇Aw mean the divergence of the column vectors w and Aw. Lastly, we make the notational conventions

$$(\hat{\cdot}) = (\cdot)/(\min_{|\xi|=1} \xi E \xi^T), \qquad N(\cdot) = \max_{V \cup \partial V} |\cdot| \qquad \text{and} \qquad N'(\cdot) = \max_{t \geqslant 0} N(\cdot).$$

We consider the solution u of the mixed initial-boundary value problem

$$Lu \equiv \nabla^{(4)}[A(\nabla^{(4)}u)^T] + (b - a_t) \cdot \nabla u + \tfrac{1}{2}(d_t - c)\, u_t = 0 \ (x \in V, t > 0),$$

$$u(x, t) = 0 \ (x \in \partial V, t \geqslant 0), \qquad u(x, 0) = f(x), \qquad u_t(x, 0) = g(x) \ (x \in V),$$

where f and g are functions in $C^1(V \cup \partial V)$ with compact support.

The plan of this paper is as follows. In Section 2 we present the basic "energy identities" upon which our work is based, and also the divergence identity from which these energy identities are derived. In Section 3 we state our main result as Theorem 1. In Section 4 we prove the Theorem using the identities of Section 2.

In its basic structure the proof is modeled on the proof of local energy decay for solutions of the wave equation given by Morawetz in [2] and [3]. Our goal is: beginning with the basic divergence identities presented in Section 2, through careful manipulation and estimation, to establish an inequality of the form

$$T^2\eta(T) - \int_{T_0}^{T} (G_1 t + G_2)\, \eta(t) \leqslant G_3\,,$$

where

$$\eta(T) = \frac{1}{2}\int_{V(T)} [e(u) + du_t^2], \qquad e(u) = \nabla u E(\nabla u)^T,$$

and $V(T)$ is the intersection of $\partial V \cup V$ with a ball, with the center at the origin, whose radius increases linearly with T.

In the case of the wave equation G_1 and G_2 are both zero, as well as the terms that give rise to them; these are the terms which cause the principal difficulty in Section 4. These terms are managed in Section 4 by a series of somewhat complex, and rather delicate estimates. In Section 5 we present a detailed comparison of our results with decay estimates obtained for the wave equation by C. S. Morawetz in [2, 3], and with the above mentioned result of E. C. Zachmanoglou.

We derive bounds for the total energy of the solutions of Problem (P) in Appendix I. In Appendix II we prove a domain of dependence theorem for the solution of Problem (P), which implies that this solution has compact support in $\partial V \cup V$ for each $t > 0$. In view of the permitted growth of total energy of the solution to Problem (P) we thought it best to present these folklore energy bounds and the related domain of dependence theorem explicitly.

2. Basic Energy Identities

The energy decay estimates of this paper are derived from the following integral identities:

$$\frac{1}{2}\int_V [du_t^2 + e(u)]_{t=T} - \frac{1}{2}\int_V [du_t^2 + e(u)]_{t=0}$$
$$= \frac{1}{2}\int_0^T \int_V [\nabla u E_t(\nabla u)^T + 2(b \cdot \nabla u)\, u_t - cu_t^2] - \int_0^T \int_V u_t L u, \qquad (2.1)$$

$$2\int_{T_0}^{T} t\int_{\partial V}(n\cdot x)(nEn^T)(n\cdot\nabla u)^2$$
$$+\frac{1}{2}\int_V\Big[(e(u)-du_r^2)(r^2+t^2)+\frac{(r+t)^2}{2r^2}d(u+ru_r+ru_t)^2$$
$$+\frac{(t-r)^2}{2r^2}d(u+ru_r-ru_t)^2\Big]_{t=T}$$
$$-\int_V[(\beta-1)r\,duu_r+2tr(a\cdot\nabla u)\,u_r+4t(a\cdot\nabla u)u]_{t=T}$$
$$-\int_V\Big\{\Big[e(u)+du_t^2+\frac{(u+2ru_T)}{r^2}du\Big]\frac{(r^2+t^2)}{2}-(\beta-1)r\,duu_r$$
$$+2tr\,du_ru_t+2t\,duu_t-2tr(a\cdot\nabla u)\,u_r-4t(a\cdot\nabla u)u\Big\}_{t=T_0}$$
$$\equiv\int_{T_0}^{T}\int_V(\nabla uE_t(\nabla u)^T+2(b\cdot\nabla u)\,u_t-cu_t^2)\frac{(r^2+t^2)}{2}$$
$$+\int_{T_0}^{T}t\int_V r[\nabla uE_r(\nabla u)^T+2(b-a_t)\cdot\nabla uu_r$$
$$+(d_t-c)\,u_ru_t+2(a_r\cdot\nabla u)\,u_t+d_ru_t^2]$$
$$-\int_{T_0}^{T}t\int_V\Big[\frac{(d_t-c)_t}{2}-\nabla\cdot(a_t-b)\Big]u^2$$
$$-2\int_{T_0}^{T}\int_V u_t[x(E-dI)(\nabla u)^T+(a\cdot x)\,u_t]$$
$$-\int_{T_0}^{T}\int_V\Big[(d_t-c)+\frac{d_t}{2}-\nabla\cdot a\Big]u^2$$
$$-\int_{T_0}^{T}\int_V[(r^2+t^2)\,u_t+2tru_t+2tu]\,Lu. \tag{2.2}$$

The above identities hold for every function $u(x, t)$ in $C^2(\text{Int } R)\cap C^1(R)$ that vanishes on $\partial V\times[0,\infty)$, and that vanishes outside a compact subset of $V\cup\partial V$ for every value of t, $t\geqslant 0$.

Equations (2.1) and (2.2) are obtained from the divergence identity

$$\nabla^{(4)}[(\alpha\cdot\nabla^{(4)}u)(Aw)-\alpha^T(\nabla^{(4)}u\,Aw)/2+u(\gamma A+B)w+C^Tu^2/2]$$
$$=[\alpha\cdot\nabla^{(4)}u+\gamma u][\nabla^{(4)}(Aw)]+(\nabla^{(4)}\cdot C)\,u^2/2$$
$$+u[C+(\nabla^{(4)}\gamma)A+\nabla^{(4)}\cdot B]w$$
$$+\nabla^{(4)}u\,\Big\{\gamma A-\frac{(\nabla^{(4)}\cdot\alpha)}{2}A-\frac{(\alpha\cdot\nabla^{(4)})A}{2}+[(\nabla^4)^T\alpha]A\Big\}\,w \tag{2.3}$$

where B is an anti-symmetric matrix, $\lambda\nabla^4 \cdot B = 0$ if $B = 0$, but if B is given by (2.4), then

$$\nabla^{(4)} \cdot B = [(\partial/\partial t)\, B^{41}, (\partial/\partial t)\, B^{42}, (\partial/\partial t)\, B^{43}, \nabla \cdot B^{*4}], \qquad w = (\nabla^{(4)}u)^T,$$

and $\nabla^{(4)}\alpha = (\alpha^j_{x_i})$.

To get (2.1) we set $\alpha = (\alpha^1, \alpha^2, \alpha^3, \alpha^4) = (0, 0, 0, 1)$, $\gamma = 0$, $B = C = 0$ in (2.3), and then integrate the resulting equation over the region bounded by the planes $t = 0$, $t = T$, and the surface $\partial V \times [0, T]$.

To get (2.2) we set $\alpha = (2xt, r^2 + t^2)$, $\gamma = 2t$,

$$C = [2t(x/r^2)d + \Delta d, -d - (t^2/r^2)d], \qquad B^{i*} = 0, \quad i = 1, 2, 3,$$
$$B^{4*} = -B^{*4} = [-(\beta + t^2/r^2)x, 0]d \tag{2.4}$$

with $(d\beta)_r + 3(d\beta)/r = 3d/r + t(d_t - c)/r + t^2 d_r/r^2$, and

$$\Delta d = 2t(b - a_t) - 2a + \{(\beta d)_t + [(t^2 d)_t]/(r^2)\}\, x - 2t(x/r^2)d.$$

We then integrate the resulting equation over the region bounded by the planes $t = T_0$, $t = T$, $T > T_0$, and the surface $\partial V \times [T_0, T]$. We follow Morawetz [3] in the manipulation of terms, which leads to (2.2) from (2.3). Victory is at hand once (2.3) has been guessed.

3. Energy Decay Estimate

The integral identities (2.1) and (2.2) imply the following.

Theorem 1. *Let $u(x, t)$ be the function that satisfies the hyperbolic equation*

$$\nabla^{(4)}[A(\nabla^{(4)}u)^T] + (b - a_t) \cdot \nabla u + \tfrac{1}{2}(d_t - c)\, u_t = 0 \tag{3.1}$$

on $V \times (0, \infty)$, that vanishes on $\partial V \times (0, \infty)$, and that satisfies the initial conditions $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$, $x \in V$, where $f(x)$ and $g(x)$ are functions of compact support in $C^1(V \cup \partial V)$. Then $u(x, t)$ has compact support in $V \cup \partial V$ for each $t > 0$ (cf. Appendix II). *Further, if Conditions* (3.3) *and* (3.4) *below hold, then there exist constants k, M, Q, T_0, and Ω (defined below) such that the inequality*

$$\frac{1}{2}\int_{V(T)} [du_t^2 + e(u)] \tag{3.2}$$
$$\leqslant \frac{\mathscr{E}(x, 0)Q}{2T^2}\left(1 + \left(\frac{T}{T_0}\right)^k e^{(M/T_0)}\left\{1 - \left(\frac{T_0}{T}\right)^k + \frac{M}{(1 + k)\, T_0}\left[1 - \left(\frac{T_0}{T}\right)^{1+k}\right]\right\}\right)$$

holds for every value of $T \geqslant T_0$ where $V(T) = \{x : r \leqslant \epsilon\Omega T\} \cap (V \cup \partial V)$.

For each $0 < \epsilon < 1$, we choose T_0 to be any positive number such that $V(T_0) \supset \partial V$. The constant Ω is defined by the equation

$$\Big(1 + 1N'(d) + N\Big(\frac{1}{r^2}\Big) N'(\hat{1})\{4N'(r^3 d_r) + 2N'[r^3(d_t - c)]\} + 4\Big[N\Big(\frac{1}{r^2}\Big)\Big]^{1/2} N'(\hat{1})\, N'(r^2 a) + 2N'(\hat{a})\Big)\, \Omega = \frac{1}{2}.$$

The constants k, M and Q in the energy decay inequality (3.2) are defined differently in each of three cases as follows.

Case 1. If for every positive number p there is some value of $t > p$ for which the quadratic form of the matrix

$$D = \begin{pmatrix} E_t & b^T \\ b & -c \end{pmatrix}$$

fails to be negative semidefinite on $V \cup \partial V$, then (3.2) holds if

$$k = [1/(1-\epsilon)](\mathrm{Max}(\alpha_1, \alpha_2) + [1 + (\epsilon\Omega)^2]\, \mathrm{Max}[\alpha_1', \alpha_2']),$$
$$M = [1/(1-\epsilon)]\, \mathrm{Max}(\alpha_3, \alpha_4),$$
$$Q = \frac{\Gamma}{(1-\epsilon)} \frac{\mathrm{Max}(\alpha_5, \alpha_6)}{(\epsilon\Omega)^{3+q}} + \frac{\Gamma}{(1-\epsilon)} \frac{\mathrm{Max}(\alpha_5', \alpha_6')}{(\epsilon\Omega)^{2+q}} \Big(1 + \frac{1}{(\epsilon\Omega)^2}\Big) + \frac{1}{(1-\epsilon)}(Q_1 + Q_2),$$

where

$$\alpha_1 = 2N'\Big(r\frac{d_r}{d}\Big) + 2N'\Big(r\frac{a_r}{d}\Big) + N'\Big[r\frac{(d_t - c)}{d}\Big],$$
$$\alpha_2 = 2N'(r\hat{E}_r) + 2N'(r\hat{a}_r) + 4N'[r(\hat{a}_t - \hat{b})] + N'[r(\hat{d}_t - \hat{c})] + 8N\Big(\frac{1}{r^2}\Big) N'(\hat{1})\, N'\Big\{r^4\Big[\frac{(d_t - c)_t}{2} - \nabla \cdot (a_t - b)\Big]\Big\},$$
$$\alpha_3 = 2N'\Big[r\Big(\frac{E}{d} - I\Big)\Big] + 4N'\Big(r\frac{a}{d}\Big),$$
$$\alpha_4 = 2N'[r(\hat{E} - \hat{d}I)] + 16N'(\hat{1})N\Big(\frac{1}{r^2}\Big) N'\Big\{r^4\Big[(d_t - c) + \frac{d_t}{2} - \nabla \cdot a\Big]\Big\},$$
$$\alpha_5 = 2N'\Big(r^{4+q}\frac{d_r}{d}\Big) + 2N'\Big(r^{4+q}\frac{a_r}{d}\Big) + N'\Big[r^{4+q}\frac{(d_t - c)}{d}\Big] + 2(\epsilon\Omega)\, N'\Big[r^{3+q}\Big(\frac{E}{d} - I\Big)\Big] + 4(\epsilon\Omega)\, N'[r^{3+q}(a/d)],$$
$$\alpha_6 = 2N'(r^{4+q}\hat{E}_r) + 2N'(r^{4+q}\hat{a}_r) + 2N'[r^{4+q}(\hat{a}_t - \hat{b})] + N'[r^{4+q}(\hat{d}_t - \hat{c})] + 8N'(\hat{1})\, N'\Big\{r^{5+q}\Big[\frac{(d_t - c)_t}{2} - \nabla \cdot (a_t - b)\Big]\Big\} + 16(\epsilon\Omega)\, N'(\hat{1})\, N'\Big\{r^{4+q}\Big[(d_t - c) + \frac{d_t}{2} - \nabla \cdot a\Big]\Big\},$$

$$\alpha_1' = N'\left(t\frac{b}{d}\right) + N'\left(t\frac{c}{d}\right),$$
$$\alpha_2' = N'(t\hat{E}_t) + N'(t\hat{b}),$$
$$\alpha_5' = N'\left(r^{4+q}\frac{b}{d}\right) + N'\left(r^{4+q}\frac{c}{d}\right),$$
$$\alpha_6' = N'(r^{4+q}\hat{E}_t) + N'(r^{4+q}\hat{b}),$$
$$Q_1 = \Big[\frac{(1+(\epsilon\Omega)^2)}{(\epsilon\Omega)^{2+s}}\left\{\frac{1}{4}N'[r^{2+s}(\hat{E}-\hat{d}I)] + N'(r^{3+s}\,d_r)\,N'(\hat{1})\right\}$$
$$+ \epsilon\Omega(N'(r^{2+s}\hat{a}) + \{2N'(r^{2+s}a) + N'[r^{3+s}(d_t - c)]\}\,N'(\hat{1})\Big]\,4\chi_1\,,$$
$$Q_2 = [4N'(\hat{1})\,(N'\{r^2[d_t(x, T_0) - c(x, T_0)]\})\,T_0 + N'[d(x, T_0)]\,T_0^2$$
$$+ N'[rd_r(x, T_0)]\,T_0^2 + 2N'[\hat{d}(x, T_0)]\,T_0 + 4N'(\hat{1})\,T_0^2 + 4T_0$$
$$+ 4N'(\hat{1})\,N'[ra(x, T_0)]\,T_0 + 4N'[r\hat{a}(x, T_0)]\,T_0$$
$$+ N'(\hat{1})\,N'[d(x, T_0)]\,T_0 r_0^2 + (r_0^2 + T_0^2)]\,\chi_2\,,$$
$$\mathscr{E}(x, 0) = \frac{1}{2}\int_V \{d(x, 0)\,g^2(x) + \nabla f(x)\,E(x, 0)[\nabla f(x)]^T\},$$
$$\chi_1 = (\Lambda/T_0^s), \qquad \chi_2 = \Lambda,$$
$$s = \operatorname{Max}\left[N'(t\hat{E}_t) + N'(t\hat{b}), N'\left(t\frac{b}{d}\right) + N'\left(t\frac{c}{d}\right)\right],$$
$$\Lambda = \left[1 + \int_0^{T_0} \operatorname{Max}\left[N(\hat{E}_t) + N(\hat{b}), N\left(\frac{b}{d}\right) + N\left(\frac{c}{d}\right)\right]\left(\frac{t}{\epsilon} + 1\right)^{s+\epsilon s'} dt\right],$$
$$s' = \operatorname{Max}\left[N(\hat{E}_t) + N(\hat{b}), N\left(\frac{b}{d}\right) + N\left(\frac{c}{d}\right)\right],$$
$$q = 1 + \delta + s, \qquad \delta > 0,$$
$$\Gamma = (\Lambda/\delta T_0^{\delta+s}),$$

and r_0 is a positive constant such that $u(x, T_0) \equiv 0$ if $r \geqslant r_0$.

The quantities defined by the above equations exist if the following conditions hold:

$$N'[r^{3+q}(E - dI)],\ N'[r^{2+s}(E - dI)],\ N'(r^{4+q}E_r) < \infty,$$
$$N'(r^{4+q}\,d_r),\ N'(r^{3+s}\,d_r),\ N'[r^{4+q}(d_t - c)],\ N'[r^{3+s}(d_t - c)] < \infty, \tag{3.3}$$
$$N'(r^{3+q}a),\ N'(r^{2+s}a),\ N'(r^{4+q}a_r),\ N'[r^{4+q}(a_t - b)] < \infty,$$
$$N'\left\{r^{4+q}\left[(d_t - c) + \frac{d_t}{2} - \nabla\cdot a\right]\right\},\ N'\left\{r^{5+q}\left[\frac{(d_t - c)_t}{2} - \nabla\cdot(a_t - b)\right]\right\} < \infty,$$

and, in addition,

$$N'(tE_t), \qquad N'(r^{4+q}E_t), \qquad N'(tb), \qquad N'(t^{4+q}c) < \infty. \tag{3.4}$$

Case 2. If the quadratic form associated with D is negative semidefinite on $V \cup \partial V$, for all $t \geqslant T_0$, then the energy decay estimate (3.2) holds with $\alpha_1' = \alpha_2' = \alpha_5' = \alpha_6' = 0$, and $s = 0$.

Case 3. If the quadratic form associated with D is negative semidefinite on $V \cup \partial V$ for all $t \geqslant 0$, then the energy decay estimate (3.2) holds with $\alpha_1' = \alpha_2' = \alpha_5' = \alpha_6' = 0$, $s = 0$, and $\Lambda = 1$.

4. Proof of the Energy Decay Estimate

The estimates that together yield the differential inequality (4.14) below, from which Theorem 1 follows, are organized in two groups. We deal separately with the terms on the right- and left-hand sides of the identity (2.2). We obtain an upper bound for the right-hand side (RHS) in terms of the total initial energy $\mathscr{E}(x, 0)$, and an integral of the energy over a local domain—the intersection of V with a ball whose radius grows linearly with T. It is this energy in a finite portion of V, call it $\mathscr{E}_{\text{loc}}(x, T)$, that we seek to prove decays as $T \to \infty$.

The LHS of (2.2) is estimated from below by $c_1 T^2 \mathscr{E}_{\text{loc}}(x, T) - c_2 \mathscr{E}(x, 0)$, where c_1 and c_2 are certain positive constants; see (4.13). This estimate is, relatively, straightforward. The proof of the main theorem is thus reduced to an easy exercise in integrating a differential inequality, namely (4.14).

We first set

$$\nabla^{(4)}[A(\nabla^{(4)}u)^T] + (b - a_t) \cdot \nabla u + \tfrac{1}{2}(d\cdot - c)\, u_t = 0$$

in (2.2), and then estimate the remaining integrals on the right-hand side of (2.2) as follows:

$$\int_{T_0}^{T} t \int_V \nabla u\, r E_r (\nabla u)^T$$

$$\leqslant N'(r\hat{E}_r) \int_{T_0}^{T} t \int_{V(t)} e(u) + \frac{1}{(\epsilon\Omega)^{3+q}} N'(r^{4+q}\hat{E}_r) \int_{T_0}^{T} \frac{1}{t^{2+q}} \int_{r \geqslant \epsilon\Omega t} e(u),$$

$$2 \int_{T_0}^{T} t \int_V r(b - a_t) \cdot \nabla u\, u_r$$

$$\leqslant 2N'[r(\hat{a}_t - \hat{b})] \int_{T_0}^{T} t \int_{V(t)} e(u)$$

$$+ \frac{1}{(\epsilon\Omega)^{3+q}} N'[r^{4+q}(\hat{a}_t - \hat{b})] \int_{T_0}^{T} \frac{1}{t^{2+q}} \int_{r \geqslant \epsilon\Omega t} e(u),$$

$$\int_{T_0}^{T} t \int_V r(d_t - c)\, u_r u_t$$

$$\leqslant \frac{1}{2} N'[r(\hat{d}_t - \hat{c})] \int_{T_0}^{T} t \int_{V(t)} e(u) + \frac{1}{2(\epsilon\Omega)^{3+q}} N'[r^{4+q}(\hat{d}_t - \hat{c})]$$

$$\times \int_{T_0}^{T} \frac{1}{t^{2+q}} \int_{r \geqslant \epsilon\Omega t} e(u) + \frac{1}{2} N' \left[r \frac{(d_t - c)}{d} \right] \int_{T_0}^{T} t \int_{V(t)} du_t^2$$

$$+ \frac{1}{2} N' \left[r^{4+q} \frac{(d_t - c)}{d} \right] \frac{1}{(\epsilon\Omega)^{3+q}} \int_{T_0}^{T} \frac{1}{t^{2+q}} \int_{r \geqslant \epsilon\Omega t} du_t^2,$$

$$2 \int_{T_0}^{T} t \int_V [r(a_r \cdot \nabla u)\, u_t + r d_r u_t^2]$$

$$\leqslant N'(r\hat{a}_r) \int_{T_0}^{T} t \int_{V(t)} e(u) + \frac{1}{(\epsilon\Omega)^{3+q}} N'(r^{4+q}\hat{a}_r) \int_{T_0}^{T} \frac{1}{t^{2+q}} \int_{r \geqslant \epsilon\Omega t} e(u)$$

$$\times \left\{ N' \left(r \frac{d_r}{d} \right) \Big/ d + N' \left(r \frac{a_r}{a} \right) \right\} \int_{T_0}^{T} t \int_{V(t)} du_t^2 + \frac{1}{(\epsilon\Omega)^{3+q}}$$

$$\times \left\{ N' \left(r^{4+q} \frac{a_r}{d} \right) + N' \left(r^{4+q} \frac{d_r}{d} \right) \right\} \int_{T_0}^{T} \frac{1}{t^{2+q}} \int_{r \geqslant \epsilon\Omega t} du_t^2,$$

$$-2 \int_{T_0}^{T} \int_V x(E - dI)(\nabla u)^T u_t$$

$$\leqslant N'[r(\hat{E} - \hat{d}I)] \int_{T_0}^{T} \int_{V(t)} e(u) + \frac{1}{(\epsilon\Omega)^{2+q}} N'[r^{3+q}(\hat{E} - \hat{d}I)] \int_{T_0}^{T} \frac{1}{t^{2+q}} \int_{r \geqslant \epsilon\Omega t} e(u)$$

$$+ N' \left[r \left(\frac{E}{d} - I \right) \right] \int_{T_0}^{T} \int_{V(t)} du_t^2$$

$$+ \frac{1}{(\epsilon\Omega)^{2+q}} N' \left[r^{3+q} \left(\frac{E}{d} - I \right) \right] \int_{T_0}^{T} \frac{1}{t^{2+q}} \int_{r \geqslant \epsilon\Omega t} du_t^2,$$

$$-2 \int_{T_0}^{T} \int_V (a \cdot x)\, u_t^2$$

$$\leqslant 2N' \left(r \frac{a}{d} \right) \int_{T_0}^{T} \int_{V(t)} du_t^2 + \frac{2}{(\epsilon\Omega)^{2+q}} N' \left(r^{3+q} \frac{a}{d} \right) \int_{T_0}^{T} \frac{1}{t^{2+q}} \int_{r \geqslant \epsilon\Omega t} du_t^2.$$

In addition, making use of the inequalities,

$$\int_{V(t)} \frac{u^2}{r^4} \leqslant 4N \left(\frac{1}{r^2} \right) N'(\hat{1}) \int_{V(t)} e(u),$$

and

$$\int_{r \geqslant \epsilon\Omega t} \frac{u^2}{r^2} \leqslant 4N'(\hat{1}) \int_{r \geqslant \epsilon\Omega t} e(u), \tag{4.1}$$

that hold for functions of compact support, we have

$$-\int_{T_0}^{T} t \int_V \left[\frac{(d_t - c)_t}{2} - \nabla \cdot (a_t - b)\right] u^2$$

$$\leqslant 4N' \left\{r^4 \left[\frac{(d_t - c)_t}{2} - \nabla \cdot (a_t - b)\right]\right\} N\left(\frac{1}{r^2}\right) N'(\hat{1}) \int_{T_0}^{T} t \int_{V(t)} e(u)$$

$$+ \frac{4}{(\epsilon\Omega)^{3+q}} N' \left\{r^{5+q} \left[\frac{(d_t - c)_t}{2} - \nabla \cdot (a_t - b)\right]\right\} N'(\hat{1})$$

$$\times \int_{T_0}^{T} \frac{1}{t^{2+q}} \int_{r \geqslant \epsilon\Omega t} e(u),$$

and

$$-2 \int_{T_0}^{T} \int_V \left[(d_t - c) + \frac{d_t}{2} - \nabla \cdot a\right] u^2$$

$$\leqslant 8N' \left\{r^4 \left[(d_t - c) + \frac{d_t}{2} - \nabla \cdot a\right]\right\} N\left(\frac{1}{r^2}\right) N'(\hat{1}) \int_{T_0}^{T} \int_{V(t)} e(u)$$

$$+ \frac{8}{(\epsilon\Omega)^{2+q}} N' \left\{r^{4+q} \left[(d_t - c) + \frac{d_t}{2} - \nabla \cdot a\right]\right\} N'(\hat{1}) \int_{T_0}^{T} \frac{1}{t^{2+q}} \int_{r \geqslant \epsilon\Omega t} e(u).$$

In Cases 2 and 3

$$\int_{T_0}^{T} \int_V [\nabla u E_t (\nabla u)^T + 2(b \cdot \nabla u)\, u_t - c u_t^2] \frac{(r^2 + t^2)}{2} \leqslant 0.$$

In Case 1 the inequality

$$\int_{T_0}^{T} \int_V [\nabla u E_t (\nabla u)^T + 2(b \cdot \nabla u)\, u_t - c u_t^2] \frac{(r^2 + t^2)}{2}$$

$$\leqslant [1 + (\epsilon\Omega)^2][N'(t\hat{E}_t) + N'(t\hat{b})] \int_{T_0}^{T} t \int_{V(t)} e(u)$$

$$+ [1 + (\epsilon\Omega)^2] \left[N'\left(t\frac{b}{d}\right) + N'\left(t\frac{c}{d}\right)\right] \int_{T_0}^{T} t \int_{V(t)} du_t^2$$

$$+ \frac{1}{(\epsilon\Omega)^{2+q}} \left(1 + \frac{1}{(\epsilon\Omega)^2}\right) [N'(r^{4+q}\hat{E}_t) + N'(r^{4+q}\hat{b})] \int_{T_0}^{T} \frac{1}{t^{2+q}} \int_{r \geqslant \epsilon\Omega t} e(u)$$

$$+ \frac{1}{(\epsilon\Omega)^{2+q}} \left(1 + \frac{1}{(\epsilon\Omega)^2}\right) \left[N'\left(r^{4+q}\frac{b}{d}\right) + N'\left(r^{4+q}\frac{c}{d}\right)\right] \int_{T_0}^{T} \frac{1}{t^{2+q}} \int_{r \geqslant \Omega\epsilon t} du_t^2$$

holds.

It follows from the preceeding estimates that if $T \geqslant T_0$, then

$$\text{RHS of (2.2)}$$
$$\leqslant \{\mathrm{Max}(\alpha_1, \alpha_2) + [1 + (\epsilon\Omega)^2] \mathrm{Max}(\alpha_1', \alpha_2')\} \int_{T_0}^{T} \frac{t}{2} \int_{V(t)} [du_t^2 + e(u)]$$
$$+ \mathrm{Max}(\alpha_3, \alpha_4) \int_{T_0}^{T} \frac{1}{2} \int_{V(t)} [du_t^2 + e(u)]$$
$$+ \left[\frac{1}{(\epsilon\Omega)^{3+q}} \mathrm{Max}(\alpha_5, \alpha_6) + \frac{1}{(\epsilon\Omega)^{2+q}} \left(1 + \frac{1}{(\epsilon\Omega)^2}\right) \mathrm{Max}(\alpha_5', \alpha_6')\right]$$
$$\times \int_{T_0}^{T} \frac{1}{t^{2+q}} \mathscr{E}(x, t), \tag{4.2}$$

where

$$\mathscr{E}(x, T) = \frac{1}{2} \int_V [du_t^2 + e(u)]_{t=T}.$$

By virtue of (2.1) (see Appendix I) we have

$$\mathscr{E}(x, t) \leqslant \Lambda \mathscr{E}(x, 0) \left(\frac{t}{T_0}\right)^s \tag{4.3}$$

for all $t \geqslant T_0$. Therefore,

$$\int_{T_0}^{T} \frac{1}{t^{2+q}} \mathscr{E}(x, t) \leqslant \Gamma \mathscr{E}(x, 0),$$

where Γ, Λ, s, q are as defined in Section 3.

We now consider the terms on the left hand side of (2.2). First of all, if ∂V is star-shaped,

$$2 \int_{T_0}^{T} t \int_{\partial V} (n \cdot x)(nEn^T)(n \cdot \nabla u)^2 \geqslant 0. \tag{4.4}$$

We set $I(T) = I_1(T) + I_2(T)$, where

$$I_1(T) = \frac{1}{2} \int_{r \geqslant \epsilon\Omega T} \Big[(e(u) - du_r^2)(r^2 + t^2)$$
$$+ \frac{(r + t)^2}{2r^2} d(u + ru_r + ru_t)^2 + \frac{(t - r)^2}{2r^2} d(u + ru_r - ru_t)^2\Big]\Big|_{t=T}, \tag{4.5}$$

and

$$I_2(T) = \frac{1}{2}\int_{V(t)} \Big[(e(u) - du_r^2)(r^2 + t^2) + \frac{(r+t)^2}{2r^2}\, d(u + ru_r + ru_t)^2 + \frac{(t-r)^2}{2r^2}\, d(u + ru_r - ru_t)^2\Big]\Big|_{t=T} - \int_V (\beta - 1) r\, du_r u\,\Big|_{t=T}. \tag{4.6}$$

It follows from (4.5) that

$$I_1(T) \geqslant \int_{r \geqslant \epsilon\Omega T} \frac{(r^2+t^2)}{2}\, \nabla u(E - dI)(\nabla u)^T\Big|_{t=T} + \frac{1}{2}\int_{r\geqslant\epsilon\Omega T} (r^2+t^2)\, d(|\nabla u|^2 - u_r^2)\Big|_{t=T},$$

which in turn implies that

$$I_1(T) \geqslant -\frac{1}{2}\left(1 + \frac{1}{(\epsilon\Omega)^2}\right)\frac{1}{(\epsilon\Omega)^s T^s}\, N'[r^{2+s}(\hat{E} - \hat{d}I)] \int_{r\geqslant\epsilon\Omega T} e(u)\,\Big|_{t=T}. \tag{4.7}$$

Recalling the definition of β given in Section 2, and making use of the fact that $r \leqslant \epsilon\Omega T$ in (4.6), we deduce first that

$$I_2(T) \geqslant \frac{(1-\epsilon\Omega)^2}{2}\, T^2 \int_{V(T)} \Big[e(u) + du_t^2 - \frac{2\epsilon\Omega}{(1-\epsilon\Omega)^2}\, du_r^2\Big]\Big|_{t=T} + \frac{(1-\epsilon\Omega)^2}{2}\, T^2 \int_{V(T)} \Big[\frac{d(u^2 + 2ruu_r)}{r^2} + \frac{u^2}{r}\, d_r\Big]_{t=T} - \frac{(1-\epsilon\Omega)^2}{2}\, T^2 \times \int_{V(T)} \frac{u^2}{r}\, d_r\,\Big|_{t=T} + \frac{T^2}{2}\int_V \frac{u^2}{r}\, d_r\,\Big|_{t=T} - \frac{T}{2}\int_V (d_t - c)\, u^2\,\Big|_{t=T} + \frac{1}{2}\int_V r\, d_r u^2\,\Big|_{t=T}. \tag{4.8}$$

By virtue of inequalities (4.1), and the identity

$$\int_{V(T)} \Big[\frac{d(u^2 + 2ru_r u)}{r^2} + \frac{u^2}{r}\, d_r\Big]\Big|_{t=T} = \int_{r=\epsilon\Omega T} d\, \frac{u^2}{r}\,\Big|_{t=T},$$

which holds for functions u that vanish on ∂V, it follows from (4.8) that

$$
\begin{aligned}
I_2(T) \geqslant{} & \frac{(1-\epsilon\Omega)^2}{2} T^2 \int_{V(T)} [e(u) + du_t^2] \Big|_{t=T} \\
& - (\epsilon\Omega) \Big\{ N'(d) + 4 \left[1 - \frac{(\epsilon\Omega)}{2}\right] N'(r^3 d_r) N\left(\frac{1}{r^2}\right) N'(\hat{1}) \\
& + 2N'[r^3(d_t - c)] N\left(\frac{1}{r^2}\right) N'(\hat{1}) + 2(\epsilon\Omega) N'(r^3 d_r) N\left(\frac{1}{r^2}\right) N'(\hat{1}) \Big\} T^2 \\
& \times \int_{V(T)} e(u) \Big|_{t=T} \\
& - \frac{1}{(\epsilon\Omega)^s T^s} \Big\{ \frac{2}{(\epsilon\Omega)^2} N'(r^{3+s} d_r) + \frac{2}{(\epsilon\Omega)} N'[r^{3+s}(d_t - c)] \\
& + \frac{1}{2} N'(r^{3+s} d_r) \Big\} N'(\hat{1}) \int_{r \geqslant \epsilon\Omega T} e(u) \Big|_{t=T} .
\end{aligned}
\tag{4.9}
$$

As for the remaining terms on the left hand side of (2.2), we have the estimate

$$
\begin{aligned}
-2 \int_V tr(a \cdot \nabla u)\, u_r \,|_{t=T} \leqslant{} & 2(\epsilon\Omega)\, T^2 N'(\hat{a}) \int_{V(T)} e(u) \Big|_{t=T} \\
& + 2 \frac{N'(r^{2+s}\hat{a})}{(\epsilon\Omega)^{1+s} T^s} \int_{r \geqslant \epsilon\Omega T} e(u) \Big|_{t=T} ;
\end{aligned}
\tag{4.10}
$$

and making use of (4.1) we deduce the estimate

$$
\begin{aligned}
-2 \int_V t(a \cdot \nabla u) u \Big|_{t=T} \leqslant{} & 4(\epsilon\Omega)\, T^2 N'(r^2 a) \left[N\left(\frac{1}{r^2}\right)\right]^{1/2} \int_{V(T)} e(u) \Big|_{t=T} \\
& + \frac{4N'(r^{2+s}a)}{(\epsilon\Omega)^{1+s} T^s} N'(\hat{1}) \int_{r \geqslant \epsilon\Omega T} e(u) \Big|_{t=T} .
\end{aligned}
\tag{4.11}
$$

Again making use of (4.1), we find that

$$
\begin{aligned}
-I(T_0) \geqslant{} & -\{4N'(\hat{1}) (T_0 N'\{r^2[d_t(x, T_0) - c(x, T_0)]\} + T_0^2 N'[d(x, T_0)] \\
& + T_0^2 N'[r\, d_r(x, T_0)]) + 2T_0 N'[\hat{d}(x, T_0)] + 4T_0^2 N'(\hat{1}) \\
& + 4T_0 + 4T_0 N'[ra(x, T_0)]\, N'(\hat{1}) + 4T_0 N'[r\hat{a}(x, T_0)] \\
& + T_0 r_0^2 N'[d(x, T_0)]\, N'(\hat{1}) + (r_0^2 + T_0^2)\} \mathscr{E}(x, T_0).
\end{aligned}
\tag{4.12}
$$

Inequalities (4.3), (4.7), (4.9), (4.10), (4.11) and (4.12) imply that

$$
\text{LHS of (2.2)} \geqslant \frac{(1-\epsilon)}{2} T^2 \int_{V(T)} [e(u) + du_t^2] - (Q_1 + Q_2)\, \mathscr{E}(x, 0). \tag{4.13}
$$

We conclude from (4.2) and (4.13) that the integral inequality

$$T^2\eta(T) - \int_{T_0}^{T} (G_1 t + G_2)\,\eta(t) \leqslant G_3 \tag{4.14}$$

holds, where

$$\eta(t) = \frac{1}{2}\int_{V(t)} [e(u) + du_t^2],$$

$$G_3 = \frac{1}{(1-\epsilon)}\Big\{Q_1 + Q_2 + \frac{1}{(\epsilon\Omega)^{3+q}}\Big[\operatorname{Max}(\alpha_5, \alpha_6)$$
$$+ (\epsilon\Omega)\left(1 + \frac{1}{(\epsilon\Omega)^2}\right)\operatorname{Max}(\alpha_5', \alpha_6')\Big]\,\Gamma\Big\}\,\mathscr{E}(x, 0),$$

$$G_1 = \frac{1}{(1-\epsilon)}\{\operatorname{Max}(\alpha_1, \alpha_2) + [1 + (\epsilon\Omega)^2]\operatorname{Max}(\alpha_1', \alpha_2')\},$$

$$G_2 = \frac{1}{(1-\epsilon)}\operatorname{Max}(\alpha_3, \alpha_4).$$

Finally, we obtain the energy decay estimate (3.2) by integrating (4.14), considering $\int_{T_0}^{T} (G_1 t + G_2)\,\eta(t)$ as the unknown.

5. Comparison of Results

In the case of the wave equation, as we mentioned in the Introduction, our estimates give a decay rate for local energy of T^{-2}, since in this case k and M in (3.2) are both zero. This is in agreement with the result first obtained by C. S. Morawetz [2, 3]. (For the wave equation in $\mathbb{R}^3$, this of course implies that energy decays locally at an exponential rate [5].)

E. C. Zachmanoglou [7] proved that energy decays locally as $T^{-1+\mu}$ $(1 > \mu \geqslant 0)$ for solutions of hyperbolic equations of the form

$$\nabla[E(x)(\nabla u)^T] - c(x)u - d(x)\,u_{tt} = 0 \tag{5.1}$$

with $c(x) \geqslant 0$ and $d(x)$ strictly positive definite. His exponent μ depends on the coefficients in (5.1) in a way similar to the dependence of our exponent k upon the coefficients in (3.1). Zachmanoglou generalized the argument used by Morawetz in [4] (in treating the wave equation) to derive his result. We followed a different approach, that of Morawetz in [3], but it turns out that Zachmanoglou's restrictions on the coefficients and ours are similar. If $E_t \equiv 0$, and $a = b = c = d_t = 0$ in equation (3.1), and $c = 0$ in equation (5.1), then these equations are the same, and our

$$k = [\max(\alpha_1, \alpha_2)]/(1-\epsilon),$$

where

$$\alpha_1 = 2N'[(rd_r)/d] \quad \text{and} \quad \alpha_2 = 2N'(r\hat{E}_r).$$

The conditions $k < 2$ and $\mu < 1$ yield the same decay rate. Now $k < 2$ if

$$N'[(rd_r)/d] < (1 - \epsilon) \quad \text{and} \quad N'(r\hat{E}_r) < (1 - \epsilon), \tag{5.2}$$

and these conditions on d and E are essentially the same as Zachmanoglou's; see [7; Eq. 12.c]. If we demand that k be less than 1, the right-hand sides in (5.2) must only be divided by 2; but in either case, $k < 2$ or $k < 1$, our method imposes the conditions (3.3) on the spatial decay of $E - dI$, E_r, and d_r. These are stronger by a factor of r than Zachmanoglou's corresponding conditions. This is the price paid for obtaining a faster rate of local energy decay.

In [1] we considered equation (3.1) with $a = b$, $c = 0$ and $d = 1$ under the hypotheses that $E_t \leqslant 0$, $c_0 \geqslant 1$, (where c_0 is defined in the Introduction) and that if $t \geqslant N$ and $r \geqslant \epsilon t + c$, then $| r\nabla E | = \mathcal{O}(t^{-2-\delta})$, $| x(E - dI)| = \mathcal{O}(t^{-1-\delta})$, and $| rE_r | = \mathcal{O}(t^{-2-\delta})$, for some positive c and δ. These are more stringent conditions than those imposed here. The methods of proof that we use here, and in [1] are similar; the estimates we use to prove Theorem 1 are much stronger.

Appendix I

In this appendix we derive the inequality (4.3) for total energy, from the energy identity (2.1).

We deal first with Case 3, $D \equiv 0$ for all $t \geqslant 0$, where D is defined by (1.1). In this case (2.1) reduces to the identity

$$\mathscr{E}(x, t) \equiv \mathscr{E}(x, 0) \tag{I-1}$$

if

$$\nabla^{(4)}[A(\nabla^{(4)}u)^T] + (b - a_t) \cdot \nabla u + \tfrac{1}{2}(d_t - c)\, u_t \equiv 0, \tag{I-2}$$

in the interior of V.

We next consider Case 2, $D \leqslant 0$ if $t \geqslant T_0$. If I-2 is satisfied, we obtain from (2.1) the identity

$$\mathscr{E}(x, T) \equiv \mathscr{E}(x, 0) + \frac{1}{2}\int_0^T \int_V [\nabla u E_t(\nabla u)^T + (b \cdot \nabla u)\, u_t - cu_t^2] \tag{I-3}$$

for $0 \leqslant T \leqslant T_0$, and the inequality

$$\mathscr{E}(x, T) \leqslant \mathscr{E}(x, 0) + \frac{1}{2}\int_0^{T_0} \int_V [\nabla u E_t(\nabla u)^T + 2(b \cdot \nabla u)\, u_t - cu_t^2], \tag{I-4}$$

for $T \geqslant T_0$.

First, it follows from (I-3) that if $0 \leqslant T \leqslant T_0$, then

$$\mathscr{E}(x, T) \leqslant \mathscr{E}(x, 0) + \int_0^T \operatorname{Max}\left(N(\hat{E}_t) + N(\acute{b}), N\left(\frac{b}{d}\right) + N\left(\frac{c}{d}\right)\right) \mathscr{E}(x, t),$$

which implies that

$$\mathscr{E}(x, t) \leqslant \mathscr{E}(x, 0)\left(1 + \frac{t}{\epsilon}\right)^{s+\epsilon s'}, \qquad \epsilon > 0, \qquad 0 \leqslant t \leqslant T_0. \tag{I-5}$$

On the other hand, it follows from (I-4) that

$$\mathscr{E}(x, T) \leqslant \mathscr{E}(x, 0) + \int_0^{T_0} \operatorname{Max}\left[N(\hat{E}_t) + N(\acute{b}), N\left(\frac{b}{d}\right) + N\left(\frac{c}{d}\right)\right] \mathscr{E}(x, t). \tag{I-6}$$

Making use of (I-5) to estimate $\mathscr{E}(x, t)$ in the integral on the right-hand side of (I-6), we obtain the result that

$$\mathscr{E}(x, T) \leqslant A\mathscr{E}(x, 0), \qquad \text{if} \quad T \geqslant T_0.$$

Finally, we consider Case 1, for each $p > 0$, $D > 0$ for some $t > p$. In this case, if u satisfies (I-2), we derive the inequality

$$\mathscr{E}(x, t) \leqslant \mathscr{E}(x, 0)\left(1 + \frac{t}{\epsilon}\right)^{s+\epsilon s'}, \qquad \epsilon > 0, \qquad 0 \leqslant t \leqslant T_0, \tag{I-7}$$

by the same argument used to get (I-5). Furthermore, it follows from (2.1) that

$$\begin{aligned}\mathscr{E}(x, T) \leqslant \mathscr{E}(x, 0) &+ \int_0^{T_0} \operatorname{Max}\left[N(\hat{E}_t) + N(\acute{b}), N\left(\frac{b}{d}\right) + N\left(\frac{c}{d}\right)\right] \mathscr{E}(x, t) \\ &+ \int_{T_0}^{T} \operatorname{Max}\left[N(\hat{E}_t) + N(\acute{b}), N\left(\frac{b}{d}\right) + N\left(\frac{c}{d}\right)\right] \mathscr{E}(x, t),\end{aligned} \tag{I-8}$$

if $T \geqslant T_0$. Using (I-7) to estimate $\mathscr{E}(x, t)$ in the first integral on the right hand side of (I-8), we obtain the integral inequality

$$\mathscr{E}(x, T) \leqslant A\mathscr{E}(x, 0) + \int_{T_0}^{T} \operatorname{Max}\left[N(\hat{E}_t) + N(\acute{b}), N\left(\frac{b}{d}\right) + N\left(\frac{c}{d}\right)\right] \mathscr{E}(x, t),$$

which implies that

$$\mathscr{E}(x, T) \leqslant A\mathscr{E}(x, 0)\left(\frac{T}{T_0}\right)^{s}, \qquad T \geqslant T_0. \tag{I-9}$$

This concludes the derivation of (4.3). We state this result as a theorem.

THEOREM 2. *Under our hypotheses on the coefficients and data of Problem (P), the total energy of its solution u at time T is bounded from above by*

$$A\mathscr{E}(x, 0)(T/T_0)^s$$

if $T \geqslant T_0$ [*this is inequality* (4.3)], *and by*

$$\mathscr{E}(x, 0)\left(1 + \frac{T}{\epsilon}\right)^{s+\epsilon s'} \qquad (\epsilon > 0)$$

if $0 \leqslant T \leqslant T_0$.

Here $\mathscr{E}(x, 0)$ is the initial energy of u. The constants A, s, s' and T_0 are as defined in Section 3.

APPENDIX II

In this appendix we prove a domain of dependence theorem for solutions of Problem (P). This theorem implies that the solutions considered in this paper have compact support in $V \cup \partial V$ for each $t > 0$. The proof is modeled on the proof given by C. Wilcox in [6] for a narrower class of equations with time independent coefficients. This fundamental result can be used to prove existence, and uniqueness of a solution of Problem (P) as in [6].

DEFINITION.

$$S_\rho(x_0) = \{x : | x - x_0 | \leqslant \rho\}, \tag{II-1}$$

and

$$C_\rho(x_0) = \{(x, t) : | x - x_0 | \leqslant \rho + p_0(T - t), t \in [0, T]\}.$$

THEOREM 3. *Under the hypothesis of Sections* 1–4 *on the coefficients of the differential operator in* (3.1), *the solution* $u(x, t)$ *of Problem* (P) *exists, and satisfies the inequality*

$$\int_{S_\rho(x_0)\cap V} [e(u) + du_t^2] \leqslant e^{HT} \int_{S_{\rho+p_0T}(x_0)} [\nabla f E(x, 0)(\nabla f)^T + d(x, 0)\, g^2] \tag{II-2}$$

for each x_0 *in* V, *where* $H > 0$ *is a constant that is independent of* $u(x, t)$.

We shall not prove existence here, but only the inequality (II-2) from which existence and uniqueness both follow.

Let

$$R_T = V \times [0, T],$$

and define

$$Fu = \nabla[E(\nabla u)^T].$$

Throughout $u(x, t)$ will denote the solution of problem (P).

LEMMA 1. *If $\varphi(x, t) \in C_0^\infty(\mathbb{R}^4)$, then*

$$\int_{R_T} u_t(Fu)\varphi = -\int_{R_T} u_t \nabla\varphi E(\nabla u)^T - \int_{R_T} \nabla u E(\nabla u_t)^T \varphi. \tag{II-3}$$

Proof. Let

$$v = \varphi E(\nabla u)^T.$$

Note that

$$\int_V [(u_t \nabla v) + (\nabla u_t v)] = 0, \tag{II-4}$$

and

$$\nabla v = \varphi(Fu) + \nabla\varphi E(\nabla u)^T.$$

Substitute for v in (II-3) and integrate over $[0, T]$.

LEMMA 2. *If $\varphi \in C_0^\infty(\mathbb{R}^4)$, then*

$$2\int_{R_T} \nabla u_t E(\nabla u)^T \varphi = \int_V e(u)\varphi \Big|_0^T - \int_{R_T} [e(u)\varphi_t + \nabla u E_t(\nabla u)^T \varphi], \tag{II-5}$$

and

$$2\int_{R_T} u_t(\nabla u_t) \cdot a\varphi = -\int_{R_T} (\nabla \cdot (\varphi a))\, u_t^2. \tag{II-6}$$

The proof is obvious.

LEMMA 3. *If $\varphi \in C_0^\infty(\mathbb{R}^4)$, then*

$$2\int_{R_T} u_t(du_{tt})\varphi = \int_V d(u_t)^2 \varphi \Big|_0^T - \int_{R_T} u_t^2(d\varphi)_t \tag{II-7}$$

for every $d \in C^1(V \cup \partial V)$.

Again, the proof is obvious. Now, using the identity (II-5) in (II-3) and subtracting the result from (II-7), we find that

$$\begin{aligned}\int_{R_T} u_t[du_{tt} - Fu]\varphi &= \frac{1}{2}\int_V du_t^2 + e(u)|_0^T \\ &\quad - \frac{1}{2}\int_{R_T} \{[du_t^2 + e(u)]\varphi_t - 2\nabla u E(\nabla\varphi)^T u_t\} \\ &\quad - \frac{1}{2}\int_{R_T} \varphi[\nabla u E_t(\nabla u)^T + d_t u_t^2].\end{aligned} \tag{II-8}$$

We now specialize the choice of φ to prove (II-2). Define

$$\psi(x) = t + \{[\rho + p_0(T - t) - | x - x_0 |]/ p_0\},$$

where p_0 is a positive constant so large that

$$p_0 \geqslant \text{Max}[N'(E/d), N'(\hat{E}), 2N'(a/d)], \tag{II-9}$$

and ρ is a fixed positive number. Set

$$\varphi(x, t) = \varphi_\delta[\psi(x) - t],$$

where $\varphi_\delta \in C_0^\infty(\mathbb{R}^1)$ and

$$\varphi_\delta(\tau) = 0 \quad \text{for} \quad \tau \leqslant -\delta, \qquad \varphi_\delta(\tau) = 1 \quad \text{for} \quad \tau \geqslant \delta,$$
$$\varphi_\delta'(\tau) \geqslant 0 \quad \text{and} \quad 0 \leqslant \varphi_\delta(\tau) \leqslant 1 \quad \text{for all} \quad \tau \in \mathbb{R}^1.$$

Note that as $\delta \to 0^+$

$$\varphi \to \chi_{C_\rho(x_0) \cap R_T} \qquad \text{(uniformly in } x \text{ and } T),$$

where χ_W is the characteristic function of the set W.

LEMMA 4. *With this choice of ψ and φ we have*

$$e(\psi) \leqslant d/N'(\hat{E}), \tag{II-10}$$

and

$$I(x, t) \equiv [du_t^2 + e(u)]\, \varphi_t - 2u_t \nabla\varphi[E(\nabla u)^T + au_t] \leqslant 0. \tag{II-11}$$

Proof. The proof of (II-10) is obvious. To get (II-11), note that

$$\varphi_t = -\varphi_\delta', \qquad \nabla\varphi = \varphi_\delta' \nabla\psi.$$

Then

$$\begin{aligned} I(x, t) &= -\varphi_\delta'\{du_t^2 + e(u) + 2u_t\, \nabla\psi[E(\nabla u)^T + au_t]\} \\ &\leqslant -[1 - p_0^{-2}N'(E/d)N'(\hat{E})(1 - 2p_0^{-1}N')(a/d)]\, \delta\text{'s}\; e(u), \end{aligned}$$

since $2(\nabla\psi \cdot a/d) \leqslant 2p_0^{-1}N'(a/d) \leqslant 1$.

Proof of II-2. Define

$$\mathscr{E}(t) = \frac{1}{2}\int_0^t \int_V [du_t^2 + e(u)]\varphi,$$

and recall that $u(x, t)$ satisfies the differential equation

$$\nabla^{(4)}[A(\nabla^{(4)}u)^T] + (b - a_t) \cdot \nabla u + \frac{[(d_t - c)\, u_t]}{2}$$
$$= \nabla[E(\nabla u)^T] + 2a \cdot \nabla u_t - du_{tt} + b \cdot \nabla u + \left(- \frac{(d_t + c)}{2} + \nabla \cdot a\right) u_t$$
$$= \nabla[E(\nabla u)^T] + \nabla \cdot (u_t a) + [a \cdot (\nabla u)]_t$$
$$- du_{tt} - \frac{(d_t + c)}{2} u_t + (b - a_t) \cdot \nabla u = 0,$$

where all the coefficients, and whichever of their derivatives that appear below, are bounded on R_∞. Then, making use of (II-6), it follows from (II-8) that

$$E'(T) = E'(0) + \frac{1}{2} \int_{R_T} [-cu_t^2 + \nabla u E_t(\nabla u)^T]\varphi$$
$$+ \int_{R_T} \varphi u_t b \cdot \nabla u + \frac{1}{2} \int_{R_T} I(x, t). \tag{II-12}$$

Making use of (II-11), it follows in turn from (II-12) that

$$\mathscr{E}'(T) \leqslant \mathscr{E}'(0) + H\mathscr{E}(T),$$

where

$$H = \mathrm{Max}[N'(c/d), N'(\hat{E}_t), N'(b/d), N'(\hat{b})].$$

Integrating this inequality, we conclude that

$$\int_V [e(u) + du_t^2]\varphi \,\Big|_{t=T} \leqslant e^{HT} \int_V [e(u) + du_t^2]\varphi \,\Big|_{t=0} .$$

Taking the limit in this inequality as $\delta \to 0^+$, we obtain (II-2).

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